DRAFT: THREE-DIMENSIONAL NONLINEAR DYNAMICS OF AN EXTENSIBLE PIPE CONVEYING FLUID

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ABSTRACT

In this paper, the nonlinear dynamics of three-dimensional (3-D) pipes conveying fluid is studied. To this aim, the fluid flow effect is modelled as a distributed load comprising the inertia, Coriolis and centrifugal forces. Taking into account the pipe extensibility and applying the Euler-Bernoulli beam theory, the coupled equations of 3-D motion are derived, using the extended Hamilton’s principle. The resulting equations are discretized via the Galerkin method and solved using the numerical Adams algorithm in the time domain. Some numerical results are obtained to illustrate the effects of some parameters, such as the flow velocity, the mass parameter, and the gravity parameter on the dynamical behaviour of the system.

INTRODUCTION

Today, the pipe conveying fluid has established itself as the new model problem for the study of stability of structures found, for example, in wide engineering applications in energy, aerospace, petroleum, and marine industries. Because of some designing criteria such as minimum weight and cost, identifying the vibrating behavior of the system is unavoidable. The linear dynamics and stability of pipes conveying fluid have been researched by many investigators; for a thorough literature survey, the reader is referred to [1].

Nonlinear dynamics of pipes conveying fluid has been researched by many researchers. For instance, Semler and Païdoussis [2] derived the nonlinear equations for planar motion of a vertical cantilevered pipe conveying fluid. Moreover, they [3] studied on the planar dynamics of a cantilevered pipe conveying fluid with an end-mass, theoretically and experimentally. In a three-part paper [5–7], the 3-D nonlinear dynamics of a vertical inextensible cantilevered pipe conveying fluid which was constrained by four nonlinear springs was considered and effects of some parameters such as different arrays of springs and end mass on the dynamics of the system were studied. The three-dimensional (3-D) dynamics of a cantilevered pipe conveying fluid subjected to arrays of four intermediate springs has been explored by Ghayesh and Païdoussis [4], theoretically and experimentally. They solved the coupled nonlinear equations of motion using a finite difference method and examined the effect of some geometrical parameters on the vibrating behaviour of the system.

To the best of authors’ knowledge, the problem of the dynamics of a cantilevered pipe conveying fluid with an extensible centreline has received very little attention. The only two studies were made very recently. Ghayesh et al. [11] investigated the nonlinear planar dynamics of a cantilevered extensible pipe carrying fluid. Moreover, Askarian et al. [12] investigated the nonlinear longitudinal-lateral dynamics of a vertical cantilevered pipe conveying pulsatile flow with an end nozzle. In the present study, the 3-D nonlinear dynamics of a cantilevered pipe conveying fluid with an extensible centreline is investigated. The fluid flow effects are modelled as a dis-
distributed load along the pipe, and the pipe structure is modelled using the nonlinear Euler-Bernoulli beam theory. The coupled nonlinear equations of motion associated with longitudinal-transverse-transverse displacements are derived via the extended Hamilton’s principle, which should be solved simultaneously. The Galerkin method and Adams algorithm are used to discretize and solve the equations of motion, respectively. Some numerical results are obtained, which show the dynamical behaviour of extensible pipes conveying fluid.

GOVERNING EQUATIONS OF MOTION

In Fig. 1(a) schematic of a vertical cantilevered pipe of length \( l \), uniform cross section \( A \), bending stiffness \( EI \) and mass per unit length of \( m \) is shown. The pipe conveys an incompressible fluid with mass per unit length \( m_f \) and flow velocity \( U_f \). The pipe is assumed to be extensible, and its longitudinal axis as well as the elastic axis and the center of mass coincide with the \( x \)-axis.

To obtain the nonlinear governing equations of motion of the pipe, the extended Hamilton’s principle is utilized, that may be expressed as

\[
\int_{t_1}^{t_2} (\delta T - \delta V + \delta W_n,c) \, dt = 0, \tag{1}
\]

where \( T \) and \( V \) are the kinetic and potential energies of the pipe structure, respectively and \( W_n,c \) denotes the virtual work of non-conservative external forces. Using Fig. 1(b) which shows the undeformed and deformed positions of an element of the pipe, one can obtain the longitudinal strain of an infinitesimal element of the pipe as follows

\[
e = \frac{ds_3 - ds}{ds} = \sqrt{(1 + u')^2 + v'^2 + w'^2} - 1. \tag{2}
\]

Thus, the strain field of the pipe in the local coordinate, \( \xi - \eta - \zeta \), can be expressed as

\[
e_{\xi\xi} = e + \zeta \rho_\eta - \eta \rho_\zeta, \quad e_{\xi\eta} = -\frac{1}{2} \zeta \rho_\xi, \quad e_{\xi\zeta} = \frac{1}{2} \eta \rho_\xi. \tag{3}
\]

Moreover, the angular velocity and curvature vectors of the pipe element are as follow

\[
\omega(s,t) = -\psi \sin \theta e_\xi + \theta e_\eta + \psi \cos \theta e_\zeta, \tag{4}
\]

\[
\rho(s,t) = -\psi' \sin \theta e_\xi + \theta' e_\eta + \psi' \cos \theta e_\zeta. \tag{5}
\]

The variation of the kinetic energy can be obtained as

\[
\int_{t_1}^{t_2} \delta T \, dt = - m_f \int_{t_1}^{t_2} \int_0^l (\ddot{u} \delta u + \ddot{v} \delta v + \ddot{w} \delta w) \, ds \, dt \\
- \int_{t_1}^{t_2} \int_0^l (J_\eta \ddot{w} \delta w' + J_\zeta \ddot{v} \delta v') \, ds \, dt, \tag{6}
\]

where the over- dot denotes the partial derivative with respect to time and \( J_\eta \) and \( J_\zeta \) are the mass moment of inertias about the \( \eta \) and \( \zeta \) axes, respectively.

The variation of the strain energy of the pipe is written as

\[
\delta U_p = \int (E \delta e_{\xi\xi} e_{\xi\xi} + \frac{1}{2} G \gamma_\xi \delta \gamma_\xi \delta \gamma_\xi + \frac{1}{2} G \gamma_\zeta \delta \gamma_\zeta \delta \gamma_\zeta) \, dv. \tag{7}
\]

Moreover, the variation of the strain energy due to gravity may be obtained as

\[
\delta U_g = (m_p + m_f)g(l - s) (\delta u' + v' \delta v' + w' \delta w'). \tag{8}
\]

The variation of the virtual work due to the fluid flow forces may be written as

\[
\delta W_{n,c} = -m_f a_f \cdot (\delta u e_x + \delta v e_y + \delta w e_z), \tag{9}
\]

where the acceleration vector of the fluid flow, \( a_f \), is as follow

\[
a_f = (\ddot{u} e_x + \ddot{v} e_y + \ddot{w} e_z + (2U_f \omega_\xi + U_f^2 \rho_\xi) e_\eta \\
+ (2U_f \omega_\eta - U_f^2 \rho_\zeta) e_\zeta. \tag{10}
\]

Based on the extended Hamilton’s principle and the fact that the energy expression holds for any instant of time, the temporal integration can be ignored with no loss of generality. Thus, by substituting Eqs. (6-8) and (9) into Eq. (1), one can obtain the governing equations of motion. The equations
of motion can be rendered dimensionless using the following parameters:

\[ \xi = \frac{s}{T}, \ \ddot{u} = \frac{\dot{u}}{T}, \ \ddot{v} = \frac{\dot{v}}{T}, \ \ddot{w} = \frac{\dot{w}}{T}, \ \beta = \frac{m_f}{m_p + m_f}, \]
\[ \sigma_n = \frac{I_n}{(m_p + m_f)l^2}, \ \Pi_0 = \frac{EA l^2}{E}, \ \gamma = \frac{m_p + m_f g l^3}{m_p + m_f}, \]
\[ \sigma_\zeta = \frac{I_\zeta}{(m_p + m_f)l^2}, \ u_f = U_f \sqrt{\frac{m_f l^2}{E}}. \quad (11) \]

**SOLUTION METHOD**

The Galerkin method is used to solve the dimensionless equations of motion. The axial and two lateral deflections of the system can be expressed as expansion of the trial functions \( \bar{u}_n, \bar{v}_n \) and \( \bar{w}_n \):

\[ \bar{u}(\xi, t) = \sum_{n=1}^{N_u} \bar{u}_n(\xi) p_n(t), \quad (12) \]
\[ \bar{v}(\xi, t) = \sum_{n=1}^{N_v} \bar{v}_n(\xi) q_n(t), \quad (13) \]
\[ \bar{w}(\xi, t) = \sum_{n=1}^{N_w} \bar{w}_n(\xi) r_n(t), \quad (14) \]

where \( p_n, q_n \) and \( r_n \) are the corresponding generalized coordinates, and the trial functions \( \bar{u}_n, \bar{v}_n \) and \( \bar{w}_n \) are assumed to be the bending mode shapes of an unloaded cantilevered pipe (beam). Substituting Eqs. (12-14) in the derived equations of motion, the dimensionless coupled equations of motion can be represented in the following matrix form:

\[ \mathbf{M} \ddot{\mathbf{A}} + \mathbf{C} \dot{\mathbf{A}} + \mathbf{K} \mathbf{A} = \mathbf{V}_{NL}, \quad (15) \]

where the non-zero submatrices are as follow

\[ \mathbf{M}_{11}^{ii} = \int_0^1 \bar{u}_i \bar{u}_i \, d\xi, \]
\[ \mathbf{M}_{12}^{ii} = \int_0^1 \bar{v}_i \bar{v}_i \, d\xi + \sigma_\zeta \int_0^1 \bar{v}_i \bar{w}_i \, d\xi, \]
\[ \mathbf{M}_{13}^{ii} = \int_0^1 \bar{w}_i \bar{w}_i \, d\xi + \sigma_\zeta \int_0^1 \bar{w}_i \bar{w}_i \, d\xi, \quad (16) \]
\[ \mathbf{C}_{22}^{ii} = 2 \sqrt{\beta} u_f \int_0^1 \bar{v}_i \bar{v}_i \, d\xi, \]
\[ \mathbf{C}_{33}^{ii} = 2 \sqrt{\beta} u_f \int_0^1 \bar{w}_i \bar{w}_i \, d\xi, \quad (17) \]

\[ \mathbf{K}_{11}^{ii} = \Pi_0 \int_0^1 \bar{u}_i \bar{a}_i \, d\xi, \]
\[ \mathbf{K}_{12}^{ii} = \int_0^1 \bar{v}_i \bar{v}_j \, d\xi + \int_0^1 \bar{v}_i \bar{v}_j \, d\xi, \]
\[ \mathbf{K}_{13}^{ii} = \int_0^1 \bar{w}_i \bar{w}_j \, d\xi + \int_0^1 \bar{w}_i \bar{w}_j \, d\xi, \quad (18) \]

\[ \mathbf{V}_{NL}^i = -2 \sqrt{\beta} u_f q_i p_j \int_0^1 \bar{v}_i \bar{v}_j \, d\xi - 2 \sqrt{\beta} u_f q_i p_j \int_0^1 \bar{v}_i \bar{v}_j \, d\xi \]
\[ - 2 \sqrt{\beta} u_f r_j \int_0^1 \bar{u}_i \bar{v}_j \, d\xi - 2 \sqrt{\beta} u_f r_j \int_0^1 \bar{u}_i \bar{v}_j \, d\xi \]
\[ + \frac{1}{2} \Pi_0 q_i q_j \int_0^1 \bar{v}_i \bar{v}_j \, d\xi - \frac{1}{2} \Pi_0 q_i q_j \int_0^1 \bar{v}_i \bar{v}_j \, d\xi \]
\[ + \frac{1}{2} \Pi_0 r_i r_j \int_0^1 \bar{u}_i \bar{v}_j \, d\xi - \frac{1}{2} \Pi_0 r_i r_j \int_0^1 \bar{u}_i \bar{v}_j \, d\xi \]
\[ - q_i q_j \int_0^1 \bar{u}_i \bar{v}_j \, d\xi - q_i q_j \int_0^1 \bar{u}_i \bar{v}_j \, d\xi, \quad (19) \]

\[ \mathbf{V}_{NL}^i = -2 \sqrt{\beta} u_f q_i p_j \int_0^1 \bar{v}_i \bar{v}_j \, d\xi - 2 \sqrt{\beta} u_f q_i p_j \int_0^1 \bar{v}_i \bar{v}_j \, d\xi \]
\[ - 2 \sqrt{\beta} u_f r_j \int_0^1 \bar{u}_i \bar{v}_j \, d\xi - 2 \sqrt{\beta} u_f r_j \int_0^1 \bar{u}_i \bar{v}_j \, d\xi \]
\[ - \frac{1}{2} \Pi_0 q_i q_j \int_0^1 \bar{v}_i \bar{v}_j \, d\xi - \frac{1}{2} \Pi_0 q_i q_j \int_0^1 \bar{v}_i \bar{v}_j \, d\xi \]
\[ - \frac{1}{2} \Pi_0 r_i r_j \int_0^1 \bar{u}_i \bar{v}_j \, d\xi - \frac{1}{2} \Pi_0 r_i r_j \int_0^1 \bar{u}_i \bar{v}_j \, d\xi \]
\[ - q_i q_j \int_0^1 \bar{u}_i \bar{v}_j \, d\xi - q_i q_j \int_0^1 \bar{u}_i \bar{v}_j \, d\xi, \quad (20) \]
\[
V_{NL}^{w} = -u_f r_p j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi - 2 \sqrt{u_f r_p j} \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi \\
- 2 \sqrt{u_f r_p j} \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi - u_f^2 r_p j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi \\
- \frac{1}{2} u_f^2 r_p j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi - u_f^2 r_p j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi \\
- \frac{3}{2} \sqrt{u_f r_p j} \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi - p r_j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi \\
+ \Pi_0 p r_j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi - 3 \sqrt{u_f r_p j} \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi \\
+ \frac{1}{2} \sqrt{u_f r_p j} \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi - q r_j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi \\
- \frac{1}{2} \Pi_0 q r_j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi - 2 r_j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi \\
+ \frac{1}{2} \Pi_0 q r_j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi - 2 r_j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi \\
+ \frac{1}{2} \Pi_0 q r_j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi - p r_j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi \\
- \frac{1}{2} \Pi_0 q r_j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi - q r_j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi \\
- q r_j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi + q r_j \int_{0}^{1} \tilde{w}_1 \tilde{w}_1' \tilde{w}_1'' d\xi. \\
(21)
\]

VALIDATION

In the present study, the numerical adaptive time-step Adams algorithm is employed to solve the equations of motion (Eq. (15)). Numerical results show that using the first 11 trial functions for longitudinal and both lateral deflections is adequate for examining the dynamics of the pipe in a wide range of flow velocities. For validation purposes, the stability boundary of the linear system is obtained for a range of mass flow parameter, \( \beta \), and compared with that given in [1]. The two sets of results are plotted in Fig. 2 where they show a great agreement. Moreover, in order to consider the validity of the nonlinear terms, the 3-D equations of motion are decomposed to two set of planar equations. The lateral bifurcation diagrams of a planar \( (u-v) \) cantilevered pipe conveying fluid is drawn and compared with those of Ref. [12] which shows the similarity results in both planes (Fig. 3). In the other hand, considering the nonlinear terms which couple the \( v \) and \( w \) degrees of freedom, it is shown that such terms are symmetric and exactly presented in both plane of vibrations.

RESULTS AND DISCUSSION

In this section, the dynamical behaviour of both horizontal and vertical pipes conveying fluid are examined, and effects of some parameters on the time response of the system are studied. To this aim, the dynamical behaviour map, power spectral density (PSD) and the tip displacement of the pipe in the plane perpendicular to the pipe axis are plotted.

Fig. 4 shows the dynamical behaviour of a horizontal cantilevered pipe conveying fluid. In particular, Fig. 4(a) shows the stability map obtained using present equations of motion for an extensible pipe, while Fig. 4(b) shows that obtained by Bajaj and Sethna using AUTO software and the inextensibility assumption [10]. As seen, the stability boundary is very similar with and without extensibility assumption. In other words, taking into account the extensibility of the centreline of the pipe does not change the stability margin nor the type of bifurcation (i.e. supercritical Hopf bifurcation). However, as seen from Fig. 4(a), according to the present theory, the pipe first undergoes planar (or 2-D) limit cycle oscillations (LCO) always past the Hopf bifurcation. This is in contrast to the results shown in Fig. 4(b) based on the inextensibility assumption. As seen from the figure, there are ranges of beta (e.g. 0.195, beta; 0.297)
where 3-D motions occur right after the bifurcation point. It is also interesting to see that according to the map shown in Fig. 4(a), 2-D LCO is the only form of motion at low values of beta; while at higher values of beta, motion becomes either 2-D (planar) or 3-D (orbital or rotary) depending on the post-critical flow velocity. Thus, one may conclude that the present theoretical model yields different results for the post-critical behaviour of the system if compared with the inextensible model. For more details, the PSD and the tip displacement of the pipe in the plane perpendicular to the pipe axis are plotted for some different values of $u_f$ at $\beta = 0.6$ (Figs. 5, 6 and 7). Results show that the flow velocity parameter has a significant effect on the type of oscillation.

The dynamical behaviour of a vertical cantilevered pipe conveying fluid is shown in Fig. 8. In particular, Fig. 8(a) shows the present results, while Fig. 8(b) shows those obtained based on the inextensible assumption [9]. These results are generally consistent with those shown in Fig. 4 for a horizontal pipe. Though it is interesting to see from Fig. 8(b) that, for some values of beta, a vertical pipe conveying fluid may undergo either 2-D or 3-D motion depending on the post-critical flow velocity, a feature not observed in Fig. 4(b). For more clarification, the PSD diagram and the top view of the pipe end displacement are shown for different values of flow velocities (Figs. 9, 10 and 11), for a cantilevered vertical pipe with $\Pi_0 = 10923.58$, $\beta = 0.6$ and $\gamma = 18.9$.

**CONCLUSION**

In this paper, the three-dimensional nonlinear flow-induced vibration of a cantilevered extensible pipe conveying fluid was studied. It was found that the pipe extensibility does
not have any influence on the stability boundary of the system. It was also found that a horizontal or a vertical extensible pipe always undergoes 2-D (planar) LCO first, past the Hopf bifurcation, but it may also perform 3-D (orbital or rotary) motion depending on the mass parameter and post-critical flow velocity.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the support given to this research by the Vali-e-Asr University of Rafsanjan.

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