DYNAMICS AND STABILITY OF A PIPE CONVEYING FLUID WITH AN IMPERFECT INLET SUPPORT

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ABSTRACT
In this paper, nonlinear dynamics of a pipe flexibly supported at the upstream end and conveying fluid is investigated. The flexible support is modelled via a cubic translational and rotational springs, simulating an imperfect support. The equation of motion is obtained via Hamilton’s principle for an open system, and the Galerkin method is used for discretizing the resulting partial differential equation. Two sets of numerical results, being representative of the dynamics of the system, are presented. They show that the system undergoes a supercritical Hopf bifurcation leading to period-1 limit cycle oscillations, similarly to a cantilevered pipe. However, at higher flow velocities, quasi-periodic and chaotic motions are observed. The amplitude of the transverse displacements is generally much higher than that for a cantilevered pipe, mainly due to the large-amplitude rigid-body motions.

INTRODUCTION
Studying dynamics and stability of pipes conveying fluid has attracted researchers’ interest for many years. This problem is presently regarded as a model problem for the study of stability of structures, along with other classical problems, such as the column subjected to a compressive load and the rotating shaft [1].

To date, most studies on the dynamics and stability of pipes conveying fluid have dealt with perfectly supported systems, such as, clamped-free (or cantilevered), pinned-pinned, and clamped-clamped pipes. There are ample examples of such studies – Refs. [2–6] to name a few. On the other hand, some studies concerned the dynamics and stability of pipes modified by additional intermediate or downstream-end flexible supports; two more recent examples of this type of studies are [7, 8]. They found that the modified systems may show very complex and sometimes surprising dynamical behaviour.

Nevertheless, there are few studies, to the best of authors’ knowledge, in which a flexible or imperfect support has been introduced to the upstream end of the pipe. Most relevant studies are [1, 9–11]. For example, stability of a pinned-free pipe conveying fluid, additionally supported at the pinned (upstream) end by a rotational spring was studied in [9]. Therein, the spring stiffness was considered to be generally flow-dependent. In [1], a general form of equation of motion was presented, whereby the dynamics of various systems with either perfect or imperfect boundary conditions can readily be investigated. Motivated by this, several studies were conducted for similar systems, such as for plates [12], and cylinders [13, 14] in axial flow.

The main objective of the present paper is to examine the nonlinear dynamics of a pipe conveying fluid flexibly/imperfectly supported at the upstream end. In particular, the linear theory presented in [1] is extended to a nonlinear theory. The imperfect end-supports are modelled via cubic translational and rotational springs.
THEORETICAL BACKGROUND

The linear equation of motion, in the variational form, for a pipe flexibly supported at its two ends and conveying fluid was presented in [1] (see Fig. 1). That is

\[
\begin{align*}
&\left[ EI w''' + MU^2 w'' + 2MU w' + (m + M) \ddot{w} \right] \delta w \\
&+ M \left[ \ddot{w} + U w' \right] \delta (s) \delta w + \left[ K_0 w \delta w + C_0 w' \delta w' \right] \delta (s) \\
&+ \left[ K_L w \delta w + C_L w' \delta w' \right] \delta (s - L) = 0,
\end{align*}
\]

(1)

where \( EI \) is the flexural rigidity of the pipe, assumed to be constant along the pipe, \( M \) is the mass of fluid per unit length, flowing steadily with velocity \( U \), \( m \) is the mass of the pipe per unit length, \( w \) is the pipe deflection in the transverse direction; \( \gamma = \partial / \partial s \) and \( \delta = \partial / \partial t \) and \( s \) and \( t \) being the curvilinear coordinate along the centreline of the pipe and time, respectively. Also, \( K_0 \) and \( C_0 \) are, respectively, translational and rotational spring constants at \( s = 0 \); \( K_L \) and \( C_L \) are the same quantities at \( s = L \); \( s \) is the curvilinear coordinate along the centreline of the pipe.

The nonlinear equation of motion for a pipe flexibly supported at its two ends and conveying fluid may be written as

\[
\begin{align*}
&\left[ (m + M) \ddot{w} + 2MU w' + (m + M) w'' + s \right] \delta w \\
&+ w' \left( MU^2 + (m + M) g (L - s) \right) + EI (1 + \frac{1}{2} w^2) \delta w \\
&+ \left[ (m + M) \ddot{w} + 2MU w' + (m + M) w'' + s \right] \delta w \\
&+ \left[ (m + M) \ddot{w} + 2MU w' + (m + M) w'' + s \right] \delta w
\end{align*}
\]

(2)

in which the translational and rotational spring constants were represented generally by \( F_i(w) \) and \( G_i(w') \) \( (i = 0, L) \), respectively, and the effects of gravity \( g \) being the gravitational acceleration and viscoelastic damping of the Kelvin-Voigt type \( (\eta \text{ being the the damping coefficient}) \) have also been taken into account.

To obtain Eq. (2), we have used Hamilton’s principle for an ‘open’ system of changing mass or a non-material volume (refer to [15, 16]) and followed derivations made in [1, 4]. Also, the centreline of the pipe is assumed to be inextensible. Equation (2) is correct to the order of \( O(\varepsilon^5) \), where \( \varepsilon < < 1 \). It should also be noted that since the upstream end of the pipe can generally translate and rotate (see Fig. 1), contrary to a clamped-free pipe, the relationship between the virtual displacements \( \delta x \) and \( \delta w \) may be written as (cf. [1]; Eq. (13), and [4]; Eq. (21))

\[
\begin{align*}
\delta x &= -w' (1 + \frac{1}{2} w^2) \delta w + w_0' (1 + \frac{1}{2} w_0^2) \delta w_0 \\
&+ \int_0^\varepsilon w'' (1 + \frac{3}{2} w^2) \delta w + O(\varepsilon^5).
\end{align*}
\]

(3)

where subscript 0 refers to \( s = 0 \).

Equation (2) may be rendered dimensionless using following quantities

\[
\begin{align*}
x &= \frac{s}{L}, \quad \eta = \frac{w}{L}, \quad \tau = \frac{EI (m + M) \varepsilon^{1/2} t}{L^2}, \quad \alpha = \frac{EI (m + M) \varepsilon^{1/2} g}{L^2}, \\
u &= \frac{m + M}{EI} \varepsilon^{1/2}UL, \quad \gamma = \frac{m + M}{LEI} L^3 g, \quad \beta = \frac{m + M}{m + M}.
\end{align*}
\]

(4)

The dimensionless equation of motion can be transformed into a set of second-order ordinary differential equations using Galerkin’s method; thus, we let

\[
\eta (\xi, \tau) = \sum_{j=1}^N \phi_j(\xi) q_j(\tau),
\]

(5)
where \( \phi_j(\xi) \) are the free-free Euler-Bernoulli beam eigenfunctions, used here as a suitable set of base functions; \( q_j(\tau) \) are the corresponding generalized coordinates and \( N \) denotes the number of eigenfunctions (modes) used in the analysis.

As previously mentioned, in this paper, the imperfect upstream end support is modeled via cubic translational and rotational springs, i.e.

\[
F_0(w) = K_1 w + K_3 w^3, \quad G_0(w') = C_1 w' + C_3 w'^3, \quad (6)
\]

where \( K_1, K_3, C_1 \) and \( C_3 \) are spring constants. The dimensionless counterpart of Eq. (6) may be written as

\[
f_0(\eta) = k_1 \eta + k_3 \eta^3, \quad g_0(\eta') = c_1 \eta' + c_3 \eta'^3, \quad (7)
\]

where \( k_1 = K_1 L^3/ EI, \quad k_3 = K_3 L^5/ EI, \quad c_1 = C_1 L/ EI, \) and \( c_3 = C_3 L/ EI. \)

The equation of motion after applying the Galerkin method takes the following form:

\[
M_{ij} \ddot{q}_j + C_{ij} \dot{q}_j + K_{ij} q_j + \alpha_{ijkl} q_k q_l + \beta_{ijkl} q_j q_k q_l + \gamma_{ijkl} \dot{q}_j \dot{q}_k q_l + \mu_{ijkl} \dot{q}_j \dot{q}_k q_l q_l = 0, \quad (8)
\]

where \( M_{ij}, C_{ij}, K_{ij} \), featuring mass, damping and stiffness matrices, respectively, and \( \alpha_{ijkl}, \beta_{ijkl}, \gamma_{ijkl}, \mu_{ijkl} \) are given in the Appendix.

To obtain the coefficients in Eq. (8), the orthonormality of the eigenfunctions, i.e. \( \delta_{ij} = \int_0^1 \phi_i \phi_j d\xi \) with \( \delta_{ij} \) being Kronecker’s delta, and the fact that \( \phi''(\xi) = \lambda^2 \phi(\xi) \), \( \lambda \) being the \( j \)th dimensionless eigenvalue of the beam, have been utilized; also, \( b_{ij} = \int_0^1 \phi_i \phi_j' d\xi, \quad c_{ij} = \int_0^1 \phi_i \phi_j'' d\xi, \quad d_{ij} = \int_0^1 \phi_i \phi_j'' d\xi. \)

**RESULTS AND DISCUSSION**

For the sake of brevity, only two sets of results, being representative of the dynamics of systems with strong stiffness nonlinearity at the upstream-end support, are presented here. The nonlinear springs at the upstream end (or inlet) are assumed to mimic an imperfect inlet support.

Figure 2 shows the bifurcation diagram for a pipe with \( \beta = 0.2, \gamma = 10, \alpha = 0, k_1 = 1, k_3 = 1, c_1 = 0.1, \) and \( c_3 = 100. \) The stiffness values correspond to a weakly nonlinear translational and a strongly nonlinear torsional spring. The ordinate in the bifurcation diagram represents the dimensionless transverse displacement of the pipe free end (i.e. \( \xi = 1 \)), and the abscissa the dimensionless flow velocity, \( u \). As seen, the pipe is stable for \( u < 3.67 \), but it loses its static stability at \( u = 3.67 \) via a supercritical Hopf bifurcation leading to period-1 limit cycle oscillations (LCO) which last until \( u = 7.4 \). The amplitude of displacements generally increases with \( u \) in this range. For \( 7.4 \leq u < 7.9 \) oscillations switch between quasi-periodic and period-1 as \( u \) is increased. For example, at \( u = 7.4 \) quasi-periodic motion is observed, while at \( u = 7.5 \) motion is period-1, and at \( u = 7.6 \) it becomes quasi-periodic again. From \( u = 7.9 \) to 8.3, quasi-periodic oscillation is the prevalent form of motion. Then for a short range (8.4 \( \leq u \leq 8.5 \)) motion becomes period-1 again. From \( u = 8.6 \) up to the maximum flow velocity investigated, chaotic motion is observed. Figures 3 to 5 show the time history, phase-plane, power spectral density (PSD), and Poincaré plots of the pipe free end displacement at \( u = 6.0, 7.4, \) and 9.0, respectively. A sequence of the pipe shape (indicated by numbers 1 to 16) is also shown for each flow velocity. In these figures, elastic modes superimposed on the rigid-body modes are evident. All the sub-figures in Fig. 3 confirm a period-1 motion at \( u = 6.0. \) Similarly, the plots in Figs. 4 and 5 confirm a quasi-periodic and a chaotic motion at \( u = 7.4 \) and 9.0, respectively.

From Fig. 2, in addition to the rich dynamics displayed by the pipe, one would notice very large values of nondimensional transverse displacement (larger than 1) at the free end of the pipe, which may seem unrealistic. However, as seen from Figs. 3 to 5 (i.e. the pipe shapes), large displacements at the free end of the pipe are mainly because the upstream end can also move and rotate – not possible for a cantilevered pipe, for example. The bifurcation diagram shown in Fig. 2 also suggests that the route to chaos is via quasi-periodic bifurcations.

Figure 6 shows the bifurcation diagram for a pipe with \( \beta = 0.2, \gamma = 10, \alpha = 0, k_1 = 0.1, k_3 = 10, c_1 = 10, \) and \( c_3 = 0. \) The stiffness values correspond to a strongly nonlinear translational and a linear torsional spring. As seen from the bifurcation diagram, the pipe is stable for \( u < 5.85 \), but it loses its static equilibrium at \( u = 5.85 \) via a supercritical Hopf bifurcation leading to period-1 LCO. The period-1 motion is observed up to \( u = 7.3 \) where it is transformed first into a period-5 motion and then into a quasi-periodic motion. From \( u = 7.6 \) to 8 the motion becomes period-1 again. Switching between period-1 and quasi-periodic motions occurs at higher flow velocities. Eventually the motion becomes chaotic at \( u = 8.5. \) As seen from the figure, the maximum amplitude of oscillations generally increases with \( u \). The bifurcation diagram in Fig. 6, similarly to the one in Fig. 2, suggests that the route to chaos is via quasi-periodic bifurcations.

Figures 7 to 9 confirm that a period-5, a quasi-periodic and a chaotic motion occur at \( u = 7.3, 8.1, \) and 8.6, respectively.

Some numerical results were also obtained for higher values of the mass ratio, \( \beta_c \), (not shown here) where the main features of the dynamics of the system shown in Figs. 2 and 6, namely the appearance of a supercritical Hopf bifurcation, switching between period-1 and quasi-steady motions, and the quasi-periodicity route to chaos are retained. However, as \( \beta \) is increased, the amplitude of displacements generally decreases. This appears to be mainly because the amplitude of the rigid-body motion considerably decreases at higher \( \beta \) values.
FIGURE 2: BIFURCATION DIAGRAM FOR A PIPE WITH $\beta = 0.2$, $\gamma = 10$, $\alpha = 0$, $k_1 = 1$, $k_3 = 1$, $c_1 = 0.1$, and $c_3 = 100$, WITH $N = 12$ MODES IN THE GALERKIN APPROXIMATION AND $\Delta \tau = 0.001$ (10$^6$ TIME-STEPS) IN THE TIME-INTEGRATION SOLUTION VIA GEAR'S BDF METHOD, SHOWING THE NONDIMENSIONAL DISPLACEMENT AT THE DOWNSTREAM END OF THE PIPE ($\xi = 1$) AS A FUNCTION OF THE DIMENSIONLESS FLOW VELOCITY, $u$.

FIGURE 3: FROM LEFT TO RIGHT: TIME HISTORY, PHASE-PLANE, POWER SPECTRAL DENSITY (PSD), Poincaré, AND PIPE SHAPE PlOTS AT $u = 6.0$, INDICATING A PERIOD-1 MOTION; $f = (\frac{1}{2\pi})(\frac{M_{en}}{E})^{1/2}\Omega^2$, $\Omega$ BEING THE DIMENSIONAL CIRCULAR (RADIAN) FREQUENCY.

FIGURE 4: FROM LEFT TO RIGHT: TIME HISTORY, PHASE-PLANE, POWER SPECTRAL DENSITY (PSD), Poincaré, AND PIPE SHAPE PlOTS AT $u = 7.4$, INDICATING A QUASI-PERIOD MOTION.

FIGURE 5: FROM LEFT TO RIGHT: TIME HISTORY, PHASE-PLANE, POWER SPECTRAL DENSITY (PSD), Poincaré, AND PIPE SHAPE PlOTS AT $u = 9.0$, INDICATING A CHAOTIC MOTION.
FIGURE 6: BIFURCATION DIAGRAM FOR A PIPE WITH \( \beta = 0.2, \gamma = 10, \alpha = 0, k_1 = 0.1, k_3 = 10, c_1 = 10, c_3 = 0, \) WITH \( N = 12 \) MODES IN THE GALERKIN APPROXIMATION AND \( \Delta \tau = 0.001 \) (10⁶ TIME-STEPS) IN THE TIME-INTEGRATION SOLUTION VIA GEAR’S BDF METHOD, SHOWING THE NONDIMENSIONAL DISPLACEMENT AT THE DOWNSTREAM END OF THE PIPE \( (\xi = 1) \) AS A FUNCTION OF THE DIMENSIONLESS FLOW VELOCITY, \( u \).

FIGURE 7: FROM LEFT TO RIGHT: TIME HISTORY, PHASE-PLANE, POWER SPECTRAL DENSITY (PSD), POINCARÉ, AND PIPE SHAPE PLOTS AT \( u = 7.3 \), INDICATING A PERIOD-5 MOTION; \( f = (\frac{1}{2\pi})(\frac{M+m}{E})^{1/2}\Omega^2 \), \( \Omega \) BEING THE DIMENSIONAL CIRCULAR (RADIAN) FREQUENCY.

FIGURE 8: FROM LEFT TO RIGHT: TIME HISTORY, PHASE-PLANE, POWER SPECTRAL DENSITY (PSD), POINCARÉ, AND PIPE SHAPE PLOTS AT \( u = 8.1 \), INDICATING A QUASI-PERIODIC MOTION.

FIGURE 9: FROM LEFT TO RIGHT: TIME HISTORY, PHASE-PLANE, POWER SPECTRAL DENSITY (PSD), POINCARÉ, AND PIPE SHAPE PLOTS AT \( u = 8.6 \), INDICATING A CHAOTIC MOTION.
CONCLUSION

It was shown in this paper that a pipe conveying fluid with a flexible upstream support (simulating the support imperfections) displays quite complex and rich dynamics including quasi-periodic and chaotic motions. The route to chaos was found to be via quasi-periodic bifurcations.

ACKNOWLEDGEMENTS

The first author appreciates the Faculty of Engineering and Computer Science of Concordia University for a Start-Up grant.

APPENDIX

\[ M_{ij} = \delta_{ij}, \quad C_{ij} = 2a\beta^{1/2}h_{ij} + u\beta^{1/2}\phi_i(0)\phi_j(0) + \alpha\lambda^4\delta_{ij}, \]

\[ K_{ij} = \lambda^4\delta_{ij} + u^2\phi_i(0)\phi_j(0) - \gamma\phi_i(0)\phi_j(0) + k_1\phi_i(0)\phi_j(0) + c_1\phi_i'(0)\phi_j'(0), \]

\[ \alpha_{ijkl} = u^2\int_0^1 \phi_i\phi_j\phi_k\phi_l\,d\xi - \frac{3}{2}\gamma\int_0^1 (1 - \xi)\phi_i\phi_j\phi_k\phi_l\,d\xi + \frac{1}{2}\gamma\int_0^1 \phi_i\phi_j\phi_k\phi_l\,d\xi + \frac{4}{3}\gamma\int_0^1 \phi_i\phi_j\phi_k\phi_l\,d\xi - \frac{3}{2}\gamma\int_0^1 \phi_i\phi_j\phi_k\phi_l\,d\xi - \frac{1}{2}\gamma\phi_i(0)\phi_j(0)\phi_k(0)\phi_l(0) + k_1\phi_i(0)\phi_j(0)\phi_k(0)\phi_l(0) + c_1\phi_i'(0)\phi_j'(0)\phi_k(0)\phi_l(0), \]

\[ \beta_{ijkl} = 2u\beta^{1/2}\int_0^1 \phi_i\phi_j\phi_k\phi_l\,d\xi - 2u\beta^{1/2}\int_0^1 \phi_i\phi_j\phi_k\phi_l\,d\xi + 2u\beta^{1/2}\phi_i(0)\phi_j(0)\phi_k(0)\phi_l(0) \]

\[ + \alpha\lambda^4\int_0^1 \phi_i\phi_j\phi_k\phi_l\,d\xi + 2\alpha\lambda^4\int_0^1 \phi_i\phi_j\phi_k\phi_l\,d\xi + 4\alpha\int_0^1 \phi_i\phi_j\phi_k\phi_l\,d\xi + 4\alpha\int_0^1 \phi_i\phi_j\phi_k\phi_l\,d\xi + 3\alpha\int_0^1 \phi_i\phi_j\phi_k\phi_l\,d\xi, \]

\[ \gamma_{ijkl} = \mu_{ijkl} = -\int_0^1 \phi_i\phi_j\int_0^1 \phi_k\phi_l\,d\xi + \int_0^1 \phi_i\phi_j\int_0^1 \phi_k\phi_l\,d\xi - \phi_i(0)\phi_j(0)\phi_k(0)\phi_l(0). \]

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