EFFICIENT FLUTTER BOUNDARY AND LIMIT CYCLE PREDICTION ALGORITHM
BASED ON THE TIME SPECTRAL METHOD

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ABSTRACT
Although traditional transonic aeroelastic numerical solution in the time domain is accurate, the high computational cost limits its application. Therefore, in this study, an efficient aeroelastic numerical solution based on the time spectral method, called Time-Spectral Fluid-Structure Interaction Method, is developed. The periodic aerodynamic forces and structure equations are calculated by the time spectral method and double iteration method, respectively. The novelty of this method is that in the inner iteration, variable with strong coupling effect is solved by the gradient method and the remaining variables are then solved by the least square method. Moreover, the computational fluid dynamics (CFD) solver is not required in the inner iteration. By using the double iteration method, not only the computational cost decrease, but the stagnation of the convergence can be avoided, which is caused by using the one-shot method to search for the critical point. The two-dimensional aeroelastic standard test cases are then used to validate this method. Results show that the flutter boundary and the critical characteristics of the limit cycle oscillation can be captured with a few iterations by the time-spectral fluid-structure interaction method, and the results of this method agree well with the results of the traditional time domain method. This method can improve 1-2 orders of magnitude in computational efficiency with a good robustness to initial conditions.

NOMENCLATURE

\( \nu \) — control volume
\( \tau \) — Pseudo time step

\( k' \) — Harmonic number
\( \omega \) — Angular frequency
\( M, K \) — Generalized mass matrix, generalized stiffness matrix
\( G \) — Structure damping matrix
\( F, q \) — Generalized forces, generalized displacements
\( V^* \) — Flutter velocity
\( k \) — Reduced frequency
\( V' \) — Flutter reciprocal velocity
\( \zeta \) — Virtual damping
\( h, \alpha \) — Plunging modes, pitching modes

INTRODUCTION
Aeroelastic problems mainly include the prediction of flutter and limit cycle oscillation (LCO) behaviors. With the increasing of aspect ratio and wing flexibility of advanced flight vehicles, these problems become more and more important. Currently, main research methodologies of flutter include flight test, wind tunnel test and computational aeroelasticity (CAE). Due to high cost and risk, flight flutter experiment and wind tunnel experiment are mainly used for later validation in the design phase. As for the CAE method coupling unsteady aerodynamics and structural dynamics, which has high accuracy, less cost and is easy to use, is widely used in the aeroelastic community.

However, the CAE is still computationally costly because it requires the employment of a fully nonlinear flow solver that
can accurately capture unsteady aerodynamic forces in the time domain. This makes it difficult to use unsteady computational fluid dynamic (CFD) solver to practical engineering applications, and has motivated the development of the aerodynamic reduced order models\(^1\)\(^-\)\(^2\) that can improve the computational efficiency of the nonlinear fluid structure interaction solution in recent years.

Except for the reduced order models, the efficient time discretization schemes based on the Fourier transformation, such as harmonic balance method and time spectral method, have also been developed to improve the computational efficiency of unsteady CFD. These schemes are suitable for the periodic flow, such as flutter boundary and LCO problems in the aeroelasticity. Hall\(^3\) first proposed the linearized Euler analyses of unsteady cascade flow to model accurately the unsteady motion of shocks in 1989. Then a nonlinear harmonic method was developed and applied to the quasi-three-dimensional inviscid Euler analysis of unsteady flows around oscillating blades by He\(^4\). Hall\(^5\) proposed the harmonic balance technique for computing complex time-periodic flows in 2002. With the harmonic balance method, the accurate CFD solutions can be solved by a small number of harmonics and the resulting equations are mathematically equivalent to steady equations. Dai\(^6\) proposed the time domain collection method and analyzed the aliasing phenomenon of the high dimensional harmonic balance method. Gopinath\(^7\) further proposed the time spectral method (TS) and deduced the concise expressions of the spectral time-derivative operator in 2005. This makes all the solutions are in the time domain and the program is easier to be modified. Then Zhan\(^8\) and Gong\(^9\) compared the accuracy and efficiency of time spectral method with the second order backward difference formula (BDF) respectively. Results show that the TS offers significantly better accuracy and efficiency than the second-order BDF method for the periodic flow with the low reduced frequency. The new GMRES/preconditioner combination is developed by Mundis\(^10\) to solve the robustness problems of TS solver with large numbers of time instances and/or large reduced frequencies. Then harmonic balance method and time spectral method have been applied to the turbomachines\(^11\), the computation of dynamic derivatives\(^12\), the simulation of helicopter rotor\(^13\), adjoint optimization\(^14\), et al and performed well.

For the flutter problems, the critical velocity, frequency and modes are unknown a priori. So it is hard for the direct application of time spectral/harmonic balance method. In fact, calculating the flutter critical velocity is equal to solving the stability problem of aeroelastic systems, where the vibration of the system becomes periodical. Therefore, the TS method is well suited for this problem because with the periodic limitation, this method establishes the balance equation to determine the flutter critical velocity, frequency and modes Thomas\(^15\) first employed the harmonic balance technique for the analysis of the transonic flutter boundary and LCO characteristics through the root-finding Newton-Raphson technique. However, for higher number of harmonics and structure degrees of freedom, the computational cost of building the Jacobian J itself is significant, thus making the Newton-Raphson method less attractive. For the transonic LCO problems of the airfoil and wing, the aeroelastic harmonic balance method was proposed by Yao\(^16\) recently. In this method, the flutter frequency is solved through the linear reduced order model, and this frequency is taken as the initial value to search the LCO frequency through the one-shot gradient method. The aerodynamic gradients are introduced to the gradient method to accelerate the convergence of the LCO frequency. Then this method was applied to analyze frequency lock-in effect during the vortex-induced vibration by Yao\(^17\). However, the robustness and the efficiency of this method are still not good enough. It still needs a good initial frequency and hundreds of iteration steps to solve the accurate LCO solutions.

Apart from calculating periodic flow by time spectral/harmonic balance method, the pseudo-spectral method has been used to calculate the non-periodic flow and aeroelastic responses recently. The hybrid backward differentiation formula/time-spectral method was proposed by Mundis\(^18\) to search the flutter boundary by solving the aeroelastic dynamic responses in different dynamic pressures. Yang\(^19\) solved the problems of non-periodic fluid-structure interaction based on the Chebyshev pseudo-spectral method. Take the computation of the flutter boundary as an example, such computing strategy still requires the calculation of the structure response with different dynamic pressures, thus the computational efficiency advantage of the pseudo-spectral method would be eliminated.

To overcome the limitation of large iteration steps and high computation cost in aeroelastic simulations, the time-spectral fluid-structure interaction method (TSFSIM) based on the double iteration method is proposed in the literature. This method is tested by the computation of subsonic/transonic flutter boundary and transonic LCO of the airfoil. Besides, estimation of the accuracy, efficiency and robustness in comparison with a typical time-marching technique is also provided.

**METHOD**

The semi-discrete form which includes the pseudo time term is as follows:

\[
V \frac{\partial U}{\partial \tau} + V \frac{\partial U}{\partial t} = -R(U)
\]

(1)

The central scheme is applied to perform spatial discretization, and node weighting coefficient method is applied to reconstruct inviscid fluxes. The implicit second order backward difference formula (BDF) is applied to perform time discretization for the traditional time domain method. The equations can be expressed as:

\[
V \frac{U^{m+1} - U^m}{\Delta \tau} + V \frac{3U^{m+1} - 4U^m + U^{m-1}}{2\Delta t} = -R(U^{m+1})
\]

(2)

The unsteady aerodynamic forces can be solved quickly by the time spectral method (TS) if the flow is periodic. The
discrete form of Euler equations by TS at the nth moment becomes:
\[
\frac{\partial}{\partial t}(V_n U_n) + \sum_{j=0}^{N-1} d_j^n V_n U_j + R_n = 0 \quad n = 0, 1, \ldots, N-1
\]  
(3)

When the number of sample points is odd, \(d_n^j\) can be expressed as:
\[
d_n^j = \begin{cases} 
\frac{1}{2}(-1)^{n-j} \csc(\frac{\pi(n-j)}{N}) & n \neq j \\
0 & n = j 
\end{cases}
\]

When the number of sample points is even, \(d_n^j\) can be expressed as:
\[
d_n^j = \begin{cases} 
\frac{1}{2}(-1)^{n-j} \cot(\frac{\pi(n-j)}{N}) & n \neq j \\
0 & n = j 
\end{cases}
\]

The robustness of TS solver would decrease with large numbers of time instances and/or large reduced frequencies\(^{[9]}\). So the GMRES algorithm with LU-SGS precondition is applied to increase the computational robustness and efficiency of the solution. Meanwhile the local time stepping and implicit residual smoothing are applied to accelerate the convergence of the solution.

Consider a generic dynamic system, the aerelastic equations can be expressed as:
\[
M \frac{d^2 q}{dt^2} + G \frac{dq}{dt} + Kq = F
\]  
(4)

The TS is applied to discretize the time derivative terms of the generalized displacements:
\[
\omega^2 \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} d_n^i j \frac{V_n}{\omega} \frac{d^2 q_i}{dt^2} + \omega \sum_{i=0}^{N-1} d_n^i M \frac{dq_i}{dt} + M \frac{d^2 q_n}{dt^2} = M \frac{d^2 q_n}{dt^2} F_n
\]  
(5)

\[
\left(\frac{\sqrt{V_n^2}}{\omega}\right)
\]

is divided by both sides of the equations, where
\[
k = \frac{\omega}{V_n}, V_n = \frac{\omega}{\sqrt{\mu}}. \quad \text{Then the formula can be expressed as:}
\]
\[
k^2 \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} d_n^i j \frac{V_n}{\omega} \frac{d^2 q_i}{dt^2} + k V_n \sum_{i=0}^{N-1} d_n^i M \frac{dq_i}{dt} + V_n^2 M \frac{d^2 q_n}{dt^2} = M \frac{d^2 q_n}{dt^2} F_n
\]  
(6)

The first periodic flow solution is performed using a prescribed body motion and the generalized force coefficients at each moment are computed by solving the formula (3), there is
\[
C = (C_t, 2C_m)^T\] for the 2-DOF airfoil flutter model. Then the flutter modes, flutter reduced frequency \(k\) and flutter reciprocal velocity \(V^*\) are updated by solving the formula (6), next return to the formula (3) to update the aerodynamic forces by the CFD solver. The iteration procedure would not stop until the residual of the balance equations converges. For the flutter analysis, the final computational results are the flutter critical velocity, frequency and modes. For the LCO analysis, the final computational results are the stable LCO frequency and modes.

We proposed a double iteration method based on the transformation of variables to be solved from the flutter velocity and the flutter frequency into the flutter reciprocal velocity and the flutter reduced frequency. For the double iteration method, the flutter reciprocal velocity is solved by the gradient method and the other variables are solved by the least square method based on the aerodynamic frozen in the inner iteration. Then the aerodynamic forces are updated in the outer iteration. At last, the virtual damping is computed by the residuals to decide the updating direction of the flutter reciprocal velocity, and the value of flutter reciprocal velocity is updated using the gradient method. The specific calculation flow chart is as follows:

**FIGURE 1: FLUTTER COMPUTATIONAL FLOW CHART**

The pitching mode is fixed for the problems of the flutter prediction. The virtual damping \(\zeta^*\) in the \(l\) th iteration is as follows, which is introduced to accelerate the convergence of the flutter reciprocal velocity:
\[
\zeta^* \left(\sum_{j=0}^{N-1} d_n^j q_j^* \right) = \text{Res}_n^l
\]  
(7)

Where \(\text{Res}_n^l\) is the residuals of equation (6) in the \(l\) th iteration and the virtual damping is solved by the least square method. The increment of the flutter reciprocal velocity \(\Delta V^*\) is solved as follows:
\[
\frac{\partial \text{Res}_n^l}{\partial V^*} \Delta V^* = \text{Res}_n^l
\]  
(8)

\[
V^*_{l+1} = V^* + \text{sgn}(\zeta^*) \Delta V^*
\]  
(9)

\[
\text{Output results, end}
\]
The flutter reciprocal velocity is updated by the formula (9). The iteration convergence of the flutter reciprocal velocity is slow in subsonic cases, so the virtual damping technique is used to accelerate the convergence:

\[ V_{itr+1} = V_{itr} - \sigma \frac{V_{itr} - V_{itr-1}}{\dot{\gamma}_{itr} - \dot{\gamma}_{itr-1}} \]  

(10)

Where \( \sigma \) is the relaxation factor, it is assigned 0.5-0.9 in order to avoid the numerical oscillation in the iteration.

The numerical simulation of the transonic LCO is similar to the flutter prediction for the solving procedure. The variable of the flutter reciprocal velocity \( V' \) is replaced with the pitching amplitude \( \alpha_m \) and the acceleration is neglected.

**NUMERICAL RESULTS**

1. **FORCED OSCILLATION TEST CASE**

The AGARD1 standard test case is chosen to validate the program, this is the two-dimensional unsteady flow test case constructed with the forced pitching oscillation of a NACA0012 airfoil. Figure 2 shows a close-up view of the unstructured grid, the total number of grid cells is 7034, the total number of nodes is 3661 and the number of nodes at airfoil surface is 268. The equations are solved by finite volume method, the central scheme is applied to perform the spatial discretization and the dual time step method is applied to perform the time integration. The spatial discretization method and the time integration method remain unchanged in the following test cases. The main flow condition and motion parameters are listed in Table 1.

**FIGURE 2: NEAR FILED GRIDS FOR THE NACA0012 AIRFOIL**

<table>
<thead>
<tr>
<th>TABLE 1: AGARD1 CASE PARAMETERS</th>
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<tbody>
<tr>
<td>Test case</td>
</tr>
<tr>
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<td>AGARD1</td>
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Table: BACT CASE STRUCTURE PARAMETERS

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<th>( a )</th>
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</table>

<table>
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<th>( b )</th>
<th>in</th>
<th>( a_f )</th>
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<td></td>
<td>0.5</td>
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<th>slug-ft²</th>
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<tr>
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<td>5.2</td>
<td>0°</td>
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</table>

The TS and BDF are applied to perform the physical time discretization. When the former is applied, 3, 5 and 7 sample points are selected in one cycle to compute the unsteady flow. When the latter is applied, each time period is divided into 120 time steps, and a total of 4 cycles are computed. The pitching moment coefficient loops of the lift and moment coefficients computed by the TS method are compared with the computational results of the BDF and the experiment value. As shown in Fig. 3, the numerical results computed by the program agree well with the experimental values which validate the accuracy of the program, and only 5 sample points are needed in one cycle for the TS method then the TS results and the BDF results agree well together. So the computational results of the flutter and LCO with current grids and program are credible.

**FIGURE 3: COMPUTATIONAL RESULTS OF TS COMPARED WITH BDF**

2. **FLUTTER PREDICTION**

The reliability of the program and grid has been validated, then for the numerical prediction of the flutter boundary, the effect of TSFSIM is studied. The two-degree-of-freedom pitch/plunge flutter of the NACA0012 airfoil is analyzed in the literature. The BACT standard test case is chosen as the computational model. The specific structure parameters are listed in Table 2.

**TABLE 2: BACT CASE STRUCTURE PARAMETERS**

<table>
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The grid in Fig. 2 is chosen for the CFD computation. As for the TSFSIM, the fixed amplitude of the pitching motion is 0.01 and there are 3 sample points of structure per period.
There are 3, 5 and 7 points of flow in one cycle by the TSFSIM respectively. The initial values of the flutter velocity and the flutter frequency are $V_f = 0.5$, $\alpha_f / \alpha_u = 0.6$. The computational results of the TSFSIM with different sample points are compared with the results of the BDF and the experimental values\(^{21}\) in the different cases \((Ma, \mu)\), as shown in Fig. 4.

![Figure 4: Comparison of the numerical results of flutter boundary](image)

FIGURE 4: COMPARISON OF THE NUMERICAL RESULTS OF FLUTTER BOUNDARY

It can be observed from the Fig. 4 that only 5 points of flow in one cycle are needed for the TSFSIM then the results of the TSFSIM agree well with the BDF results. The TSFSIM has been used for this case in combination with 3, 5 and 7 points of flow per period then the comparison of the efficiency are shown in Fig. 5.

![Figure 5: Efficiency of the TSFSIM compared with the BDF](image)

FIGURE 5: EFFICIENCY OF THE TSFSIM COMPARED WITH THE BDF

For the numerical prediction of the subsonic flutter boundary, the efficiency is improved by the TSFSIM but the improvement is less than one order of magnitude. But for the numerical prediction of the transonic flutter boundary, there is nearly two orders of magnitude improvement in the computational efficiency by the TSFSIM, which shows the significant effect.

The robustness of the TSFSIM is also validated in the literature. For the \(Ma = 0.6\) state, the different initial values of the flutter velocity, frequency and modes are chosen and the computational results are shown in Fig. 6. It can be seen from Fig. 6 that the final flutter velocity $V_f$ and flutter frequency $\alpha_f / \alpha_u$ are largely independent of the initial values.

![Figure 6: Evolution of the flutter velocity and frequency for various initial values](image)

FIGURE 6: EVOLUTION OF THE FLUTTER VELOCITY AND FREQUENCY FOR VARIOUS INITIAL VALUES

3. TRANSONIC LCO SIMULATION

For the state of \(Ma = 0.8\), the transonic LCO responses of the BACT model with different dynamic pressures are solved by the TSFSIM. The structure parameters are shown in Table 2 and the computational grid is shown in Fig. 2.

There are 5 sample points of flow per period, 3 and 5 sample points of structure are selected per period in the TSFSIM. The initial frequency is $\alpha_f / \alpha_u = 0.6$. The initial mode amplitude is 0.02. The comparison of the TSFSIM results and the BDF results is shown in Fig. 7.

![Figure 7: Comparison of the LCO amplitude between the TSFSIM and the BDF](image)

FIGURE 7: COMPARISON OF THE LCO AMPLITUDE BETWEEN THE TSFSIM AND THE BDF

It can be seen from Fig. 7 that the accuracy of the TSFSIM would be low if there are still 3 sample points of structure per period in the method for the nonlinear characteristics of the transonic LCO. And the results of the TSFSIM agree well with the BDF results when the number of sample points of structure is 5 in the TSFSIM.

By using the accurate solution of the previous iteration step as the initial values of the current iteration when solving the
TSFSI-CFD, the computational time is 1-3 hours. But the computation of transonic LCO by the BDF needs 80-100 hours. So there are two orders of magnitude improvement in the computational efficiency by the TSFSIM compared with the BDF.

CONCLUSION

An efficient aeroelastic solver is developed by using the time-spectral fluid-structure interaction method based on the double iteration method in the paper. For the numerical prediction of the flutter boundary, the flutter velocity, frequency and modes can be solved accurately by the TSFSIM. And there is 1-2 orders of magnitude improvement in the computational efficiency compared with the typical aeroelastic time-marching technique. For the transonic LCO problems, results show that the TSFSIM is expected to compute LCO conditions two orders of magnitude faster than the typical aeroelastic time-marching technique. This method also has good robustness, which is largely independent of the initial values.

ACKNOWLEDGMENTS

REFERENCES